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THE MEAN VALUE OF THE AREA OF POLYGONS CIRCUMSCRIBED ABOUT A PLANE CONVEX BODY

BY
J. R. SANGWINE-YAGER

ABSTRACT

Inner parallel bodies are used to prove that the mean area of polygons circumscribed about a convex body K of given area is minimum when K is a circle.

1. Introduction

In this paper, by a *convex body* we shall mean a compact convex subset of the Euclidean plane E^2 with non-empty interior. The area of a convex body K shall be denoted by $A(K)$, the perimeter by $L(K)$. A *polygon* in E^2 is a convex body which is the intersection of finitely many closed halfplanes. The definitions and important properties of convex bodies and polygons to be used in this paper can be found in [8] and [9]. A polygon P is said to be circumscribed about a convex body K if $K \subset P$ and each edge of P lies in a supporting line of K. Let P be a polygon circumscribed about the unit ball B in E^2. For each θ, $0 \leqq \theta < 2\pi$, let $P(K, \theta)$ be the polygon circumscribed about the convex body K with edges parallel to the rotation of P about the origin by the angle θ in the clockwise direction. The main result of this paper is the inequality

$$\int_0^{2\pi} A(P(K,\theta))d\theta \geqq 2A(P)A(K), \tag{1}$$

where equality holds if and only if K is a ball in E^2. The techniques we develop for the proof should be of independent interest.

If P is a square circumscribed about B, then the left-hand side of (1) is proportional to the mean value of the areas of rectangles circumscribed about K.

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For this special case, the result was first proved by Radziszewski [13], and rediscovered by Chernoff [7]. There are generalizations by Heil [11]. Similar results in higher dimensions, for P an n-dimensional cube, have been found by Chakerian [4] and [5], and Schneider [14]. These results have recently been generalized by Lutwak [12].

In the next section we establish some properties of inner parallel bodies and their form bodies. These results will be used in Section 3 to prove (1).

2. Form bodies and parallel bodies

The definition and properties of the polar dual of a convex body can be found in Eggleston [8], along with the definition of extreme points of the boundary of a convex body. We shall be concerned with the duals of extreme boundary points, namely extreme supporting lines.

For each unit vector $u \in S^1$ the value of the *supporting function* $h(K, u)$ of a convex body K, in the direction u, is the signed distance from the origin 0 to the supporting line with outer normal u. The following properties of the supporting function can also be found in [8, pp. 47–55 and p. 80]:

$$h(K, \varphi_1 u_1 + \varphi_2 u_2) \leqq \varphi_1 h(K, u_1) + \varphi_2 h(K, u_2), \qquad \varphi_1, \varphi_2 > 0 \tag{2}$$

and

$$h(K_1 + K_2, u) = h(K_1, u) + h(K_2, u),$$

where K, K_1 and K_2 are convex bodies and u, u_1 and u_2 are unit vectors. The supporting function is monotone under set inclusion. That is, if $K_1 \subset K_2$, then

$$h(K_1, u) \leqq h(K_2, u),$$

for all u in S^1. It is also a continuous function of u.

Suppose K is a convex body with the origin in its interior. A supporting line to K with outer normal u is said to be an *extreme supporting line* if it is not possible to find two distinct lines L_1 and L_2, say

$$L_i = \{x : \langle x, a_i \rangle = 1\}, \qquad i = 1, 2, \tag{3}$$

which do not intersect the interior of K and are such that

$$\frac{u}{h(K, u)} = \alpha a_1 + (1 - \alpha) a_2, \qquad 0 < \alpha < 1. \tag{4}$$

Notice that we require $h(K, u) \neq 0$. In general, we shall call a supporting line to a convex body extreme, if the supporting line with the same outer normal is an extreme supporting line to a translate of K which has the origin as an interior point. The next lemma gives a useful, equivalent definition of an extreme supporting line.

LEMMA 1. *Let K be a plane convex body and $u \in S^1$.*

The supporting line to K with outer normal u is extreme if and only if for any distinct unit vectors u_1 and u_2, and scalars φ_1 and φ_2 such that

$$u = \varphi_1 u_1 + \varphi_2 u_2, \qquad \varphi_1, \varphi_2 > 0, \tag{5}$$

inequality holds in (2). *That is,*

$$h(K,u) < \varphi_1 h(K,u_1) + \varphi_2 h(K,u_2). \tag{6}$$

PROOF. It suffices to prove the lemma for convex bodies with the origin as an interior point. Suppose

$$L = \{x : \langle x, u\rangle = h(K,u)\}$$

is an extreme supporting line of K. Choose unit vectors u_1 and u_2, as in (5). Define

$$\alpha = \varphi_1 h(K,u_1)/(\varphi_1 h(K,u_1) + \varphi_2 h(K,u_2))$$

and

$$a_i = (\varphi_i/((\alpha(2-i) + (1-\alpha)(i-1))h(K,u)))u_i, \qquad i = 1,2.$$

Then $0 < \alpha < 1$ and (4) holds. Since u is extreme, either L_1 or L_2, in (3), intersects the interior of K. That is, for $i = 1$ or $i = 2$,

$$(\alpha(2-i) + (1-\alpha)(i-1))h(K,u)/\varphi_i < h(K,u_i).$$

This expression is equivalent to (6).

On the other hand, choose distinct points a_1 and a_2, and scalar α, as in (4). For $i = 1,2$, let

$$u_i = a_i/\|a_i\|$$

and

$$\varphi_i = (\alpha(2-i) + (1-\alpha)(i-1))h(K,u)\|a_i\|. \tag{7}$$

Substituting (7) into (6) yields

$$1 < \alpha\|a_1\|h(K,u_1) + (1-\alpha)\|a_2\|h(K,u_2).$$

Therefore, for $i=1$ or $i=2$, $\|a_i\|h(K,u_i)>1$. But, this implies that the corresponding line L_i intersects K.

Let $U(K)\subset S^1$ denote the set of extreme directions of K. That is, $u\in U(K)$ if and only if the supporting line to K with outer normal u is extreme. The set $U(K)$ is never empty. In fact, every convex body is the intersection of the closed halfplanes, containing it, which are bounded by its extreme supporting lines. This result is the dual of the Krein–Milman Theorem and the result may be found in Eggleston [8, p. 27]. In our notation, the result becomes

$$K=\bigcap_{u\in U(K)}\{x:\langle x,u\rangle\leqq h(K,u)\}.$$

We shall refer to these closed halfplanes as extreme supporting halfplanes.

The concept of extreme supporting lines can be extended to higher dimensions. The next three lemmas, however, are only valid in E^2.

Lemma 2. *Let K be a plane convex body. The set of extreme directions of K, $U(K)$, is closed.*

Proof. Suppose $u\in S^1$ is not an extreme direction of some plane convex body K. There are unit vectors u_1 and u_2, as in (5), for which equality holds in (2). Let u_3 be any other unit vector which is a positive linear combination of u_1 and u_2, and distinct from u. That is,

$$u_3=\gamma_1u_1+\gamma_2u_2,\qquad \gamma_1,\gamma_2>0.$$

Since we are in E^2, the vector u may be expressed as a positive linear combination of u_1 and u_3 or u_2 and u_3. Suppose

$$u=\psi_1u_1+\psi_2u_3,\qquad \psi_1,\psi_2>0.$$

Then

$$\begin{aligned}h(K,u)&\leqq\psi_1h(K,u_1)+\psi_2h(K,u_3)\\&\leqq(\psi_1+\psi_2\gamma_1)h(K,u_1)+\psi_2\gamma_2h(K,u_2)\\&=h(K,u).\end{aligned}$$

Equality must hold throughout. Therefore, u_3 is not an extreme direction of K.

We have shown that the set of non-extreme directions of K is open. Hence the complement, $U(K)$, is closed.

Lemma 3. *Let K_1 and K_2 be two plane convex bodies. Then*

$$U(K_1+K_2)=U(K_1)\cup U(K_2).$$

PROOF. Suppose $u\in U(K_1)$ and u_1 and u_2 are distinct unit vectors as in (5). The inequality in (6) is valid for $K=K_1$ and

$$h(K_1+K_2,u)=h(K_1,u)+h(K_2,u)$$
$$<\varphi_1 h(K_1+K_2,u_1)+\varphi_2 h(K_1+K_2,u_2).$$

Therefore, by Lemma 1, $u\in U(K_1+K_2)$. Similarly, $U(K_2)\subset U(K_1+K_2)$.

On the other hand, suppose $u\notin U(K_1)$ and $u\notin U(K_2)$. Since the set of non-extreme directions is open, there are distinct vectors u_1 and u_2, as in (5), such that equality holds in (2) for $K=K_1$ and $K=K_2$. Therefore, $u\notin U(K_1+K_2)$.

This completes the proof.

In higher dimensions it is possible for the Minkowski sum of two polytopes to have a face which is not parallel to a face of either polytope. One can only show that

$$U(K_1)\cup U(K_2)\subset U(K_1+K_2).$$

The form body of a convex body K is defined by

$$K^*=\bigcap_{u\in U(K)}\{x:\langle x,u\rangle\leqq 1\}.$$

If K is a polygon, then K^* is the polygon circumscribed about the unit ball with edges parallel to K.

LEMMA 4. *Let K be a convex body. Then*

$$U(K)=U(K^*).$$

PROOF. To prove the lemma we shall use the dual of the following result of Grunbaum [9, p. 18]. If a convex body K can be expressed as the convex hull of any set $A\subset E^2$, then the set of extreme points of K is contained in A.

Suppose K is a convex body with the origin in its interior, A is a closed subset of S^1 and $f(u)$ is a continuous positive function on S^1, such that

$$K=\bigcap_{u\in A}\{x:\langle x,u\rangle\leqq f(u)\}.$$

We shall show that the set of extreme directions of K is a subset of A, and that for $u\in U(K)$, $f(u)=h(K,u)$. To verify this, recall that the polar dual of the closed halfspaces above are the line segments from the origin to $u/f(u)$. The polar dual of K is the convex hull of the closure of $C = \{u/f(u): u \in A\}$ and the origin, see [8, p. 25]. Since A is a closed set and $f(u)$ is a continuous function of u, C is also a closed set. Therefore, the polar dual of K is the convex hull of $C \cup \{0\}$, and the set of extreme points of the dual is a subset of C. It was stated above that extreme boundary points and supporting lines are dual. That is, $u \in U(K)$ if and only if $u/h(K,u)$ is an extreme point of the dual of K. This implies

$$\{u/h(K,u): u \in U(K)\} \subset C.$$

Therefore, $U(K) \subset A$ and if $u \in U(K)$, $f(u) = h(K,u)$.

Referring to the definition of K^* and Lemma 2, we have $U(K^*) \subset U(K)$.

To prove the other inclusion, choose $u \in U(K)$. Then

$$B \subset K^* \subset \{x : \langle x, u \rangle \leqq 1\},$$

and $h(K^*, u) = 1$. Again, choose unit vectors u_1 and u_2 as in (5). The supporting function of K^* in the directions of u_1 and u_2 is greater than or equal to one, and the sum $\varphi_1 + \varphi_2$ exceeds 1. Therefore,

$$\begin{aligned} h(K^*, u) = 1 &< \varphi_1 + \varphi_2 \\ &\leqq \varphi_1 h(K^*, u_1) + \varphi_2 h(K^*, u_2). \end{aligned}$$

Using Lemma 1 once more, we have shown that u is also an extreme direction of K^*.

The *inner parallel body* of a convex body K at a distance $\lambda \geqq 0$ is defined to be the set of points of K whose distance from the boundary of K is at least λ. The *inradius* r is the radius of the largest ball contained in K. The inner parallel body at a distance $r - \lambda$, $0 \leqq \lambda \leqq r$, shall be denoted K_λ. The *kernel* K_0 is the locus of centers of balls of radius r contained in K, and $K = K_r$. The inradius of an inner parallel body K_λ, $0 \leqq \lambda \leqq r$, is λ. The inner parallel body of K_λ at a distance $\lambda - \delta$, $0 \leqq \delta \leqq \lambda$, is the inner parallel body of K at the distance $r - \delta$. That is, $(K_\lambda)_\delta = K_\delta$. For all λ, $0 < \lambda \leqq r$, the set of extreme directions of K_λ is $U(K_\lambda)$ and the form body of K_λ shall be denoted by K_λ^*. An equivalent definition of the inner parallel body is that K_λ can be expressed as the intersection of the supporting halfspaces of K translated inward a distance $r - \lambda$. That is,

$$K_\lambda = \bigcap_{u \in S^1} \{x : \langle x, u \rangle \leqq h(K,u) - (r - \lambda)\},$$

and therefore, for all $u \in S^1$,

$$h(K_\lambda, u) \leqq h(K, u) - (r - \lambda).$$

As the supporting halfspaces move inward, the parallel bodies become less smooth. For example, the edges of the inner parallel bodies of a polygon are parallel to those of the original polygon, but the inner bodies may have fewer edges. With the exception of convex bodies which are the Minkowski sum of their kernel and a ball, the inner parallel bodies of all smooth convex bodies will eventually develop exposed points. The inner parallel bodies of an ellipse, close to the kernel, are lenses. The next lemma verifies this observation.

LEMMA 5. *Let K be a plane convex body with inradius r. For all λ, $0 < \lambda \leqq r$,*

$$U(K_\lambda) \subset U(K).$$

PROOF. The inner parallel body K_λ is the intersection of translates of supporting hyperplanes of K. This intersection is taken over S^1. Using the result developed in the proof of Lemma 4, it follows that the set of extreme supporting halfplanes of K_λ must be a subset of the set of translated supporting halfplanes of K. That is, for $u \in U(K_\lambda)$

$$\{x : \langle x, u \rangle \leqq h(K_\lambda, u)\} = \{x : \langle x, u \rangle \leqq h(K, u) - (r - \lambda)\}$$

or

$$h(K_\lambda, u) = h(K, u) - (r - \lambda).$$

For $u \in U(K_\lambda)$, choose u_1 and u_2 as in (5). Then

$$\begin{aligned} h(K, u) &= h(K_\lambda, u) + (r - \lambda) \\ &< \varphi_1 h(K_\lambda, u_1) + \varphi_2 h(K_\lambda, u_2) + (r - \lambda) \\ &\leqq \varphi_1 h(K, u_1) + \varphi_2 h(K, u_2) + (1 - \varphi_1 - \varphi_2)(r - \lambda) \\ &< \varphi_1 h(K, u_1) + \varphi_2 h(K, u_2). \end{aligned}$$

Therefore, $u \in U(K)$.

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be an increasing sequence of positive numbers bounded above by r. From the definition of the form body and Lemma 5 we have

$$K^*_{\lambda_1} \supset K^*_{\lambda_2} \supset \cdots \supset K^*_{\lambda_n} \supset K^*. \tag{8}$$

We shall now prove the theorem needed for the next section.

THEOREM. *Let $K \subset E^2$ be a convex body with inradius r. For all λ, $0 < \lambda < r$,*

$$K_\lambda + (r-\lambda)K^* \subset K \subset K_\lambda + (r-\lambda)K^*_\lambda. \tag{9}$$

PROOF. Choose λ, $0<\lambda<r$. From Lemmas 3, 4 and 5

$$U(K_\lambda + (r-\lambda)K^*) = U(K).$$

For all $u \in U(K)$, we have $h(K^*, u) = 1$, and

$$h(K_\lambda + (r-\lambda)K^*, u) \leqq h(K, u).$$

The left-hand inclusion in (9) now follows from the dual of the Krein–Milman Theorem. The right-hand inclusion follows in a similar manner. In this case

$$U(K_\lambda + (r-\lambda)K^*_\lambda) \subset U(K),$$

and, for all $u \in U(K)$,

$$h(K_\lambda + (r-\lambda)K^*_\lambda, u) \geqq h(K_\lambda + (r-\lambda)K^*, u) = h(K, u).$$

It should be noted that in E^n, $n>2$, only the left-hand inclusion in (9) is valid for all convex bodies. For example, if K is a polytope in E^3 and K^*_λ is not analogous, in the sense of Aleksandrov [1, p. 1205], to K_λ, then the sum of the two polytopes may have an additional face and be properly contained in K. This is the difficulty which was mentioned after the proof of Lemma 3.

It is shown in Hadwiger [10, p. 149] that the K_λ, $0 \leqq \lambda \leqq r$, form a concave family of sets. That is, if $0 \leqq \alpha \leqq 1$ and $0 \leqq \lambda_1 < \lambda_2 \leqq r$, then

$$K_{(1-\alpha)\lambda_1+\alpha\lambda_2} \supset (1-\alpha)K_{\lambda_1} + \alpha K_{\lambda_2}.$$

It follows from the properties of the supporting function that $h(K_\lambda, u)$, for fixed u, is a concave function of λ on the interval $0 \leqq \lambda \leqq r$. Hence, the right- and left-hand derivatives of $h(K_\lambda, u)$, with respect to λ, exist everywhere and are equal except on a countable set. We shall compute the derivative of $h(K_\lambda, u)$ as a corollary to the theorem.

COROLLARY 1. *Let K be a plane convex body with inradius r. For each $u \in S^1$, $h(K_\lambda, u)$ is an almost everywhere differentiable function of λ, $0<\lambda<r$, and when it exists*

$$\frac{d}{d\lambda} h(K_\lambda, u) = h(K^*_\lambda, u).$$

Furthermore,

$$\frac{d}{d\lambda} h(K_\lambda, u) \geqq 1. \tag{10}$$

Equality holds for almost all λ, $0<\lambda<r$, *and for all* $u \in S^1$ *if and only if* $K = K_0 + rB$.

PROOF. By the comments made above, it is sufficient to evaluate the right-hand derivative of $h(K_\lambda, u)$. Choose real numbers λ and ε, where $0<\lambda<r$ and $0<\varepsilon<r-\lambda$. If we replace K with $K_{\lambda+\varepsilon}$, the expression in (9) becomes

$$K_\lambda + \varepsilon K^*_{\lambda+\varepsilon} \subset K_{\lambda+\varepsilon} \subset K_\lambda + \varepsilon K^*_\lambda. \tag{11}$$

Let u be a unit vector. Using the properties of the supporting function discussed above, we have

$$h(K^*_{\lambda+\varepsilon}, u) \leqq \frac{h(K_{\lambda+\varepsilon}, u) - h(K_\lambda, u)}{\varepsilon} \leqq h(K^*_\lambda, u).$$

From (8) $h(K^*_\lambda, u)$ is a monotone function of λ, for u fixed, and as such can have at most a countable number of discontinuities. It follows from (11) that the right-hand derivative of $h(K_\lambda, u)$ is equal to $h(K^*_\lambda, u)$, except at its points of discontinuity.

Since $K^*_\lambda \supset B$, $h(K^*_\lambda, u) \geqq 1$. To prove the necessary and sufficient conditions for equality, define

$$c(\lambda, u) = h(K_\lambda, u) - h(K_0, u) - \lambda, \qquad 0<\lambda<r, \quad u \in S^1.$$

This is an absolutely continuous functions of λ. If equality holds in (10), for almost all λ, and for all $u \in S^1$, then $\partial c/\partial\lambda = 0$, for almost all λ. Since $c(0,u)=0$, the function $c(\lambda, u)$ is identically zero. In particular,

$$c(r,u) = h(K,u) - h(K_0,u) - r = 0.$$

Hence $K = K_0 + rB$.

The results of the next corollary were first proved by Bol [2] to verify the sharpened version of the isoperimetric inequality

$$L^2(K) \geqq 4A(K^*)A(K).$$

Equality holds in this inequality if and only if K and K^* are homothetic.

The *mixed area* of two plane convex bodies K_1 and K_2 shall be denoted $A(K_1, K_2)$. The definition of properties of mixed areas, which we shall need, can be found in [8, pp. 82–89]. In particular,

$$A(K) = A(K,K) \quad \text{and} \quad A(K,B) = \tfrac{1}{2}L(K).$$

Suppose K_1 is a convex body and K_2 is a polygon with m edges. Let u_i, $1 \leqq i \leqq m$, be the unit outer normal to the ith edge and l_i the length of this edge. Eggleston [8, p. 85] shows that

$$A(K_1, K_2) = \frac{1}{2}\sum_{i=1}^{m} h(K_1, u_i)l_i. \tag{12}$$

The following integral representation for the mixed area of any two convex bodies K_1 and K_2 can be derived from (12), by a limiting process:

$$A(K_1, K_2) = \frac{1}{2}\int h(K_1, u)ds(K_2, u).$$

The integration is with respect to the arc length measure of K_2. Notice that this representation may be used to show that the perimeter of convex body K is twice its mixed area with the unit ball B by letting $K_1 = B$ and $K_2 = K$. Recall, that for $u \in U(K), h(K^*, u) = 1$. Therefore, if we let $K_1 = K^*$ and $K_2 = K$, we also have

$$2A(K, K^*) = L(K).$$

Furthermore, $A(K_\lambda, B)$ and $\sqrt{A(K_\lambda, K_\lambda)}$ are concave monotone functions of λ. The techniques used in the proof of the first corollary may be employed to prove the next.

Corollary 2. *Let K be a plane convex body with inradius r. Then $L(K_\lambda)$ and $A(K_\lambda)$ are almost everywhere differentiable functions of λ, $0 < \lambda < r$, and when their derivatives exist*

$$\frac{d}{d\lambda}L(K_\lambda) = L(K^*_\lambda)$$

and

$$\frac{d}{d\lambda}A(K_\lambda) = L(K_\lambda).$$

3. Proof of the main result

Suppose J and J' are continuous real valued functionals defined for all convex bodies in E^n. Let $K \subset E^n$ be a convex body with inradius r. Suppose further that $J(K_\lambda)$ is absolutely continuous, for $0 < \lambda < r$, and that

$$\frac{d}{d\lambda}J(K_\lambda) \geqq J'(K_\lambda), \qquad 0 < \lambda < r. \tag{13}$$

Then

$$J(K) \geqq J(K_0) + \int_0^r J'(K_\lambda)\,d\lambda, \tag{14}$$

and equality holds if and only if equality holds in (13) almost everywhere for $0 < \lambda < r$.

This observation was employed by Bol [2] and [3] to prove the isoperimetric inequality mentioned at the end of Section 2. For various choices of J and J', Chakerian [4], [5] and [6], has also proved several geometric inequalities. It should be noted that the conditions on J and J' differ from [4] and [5]. We have required J to be an absolutely continuous function on the inner parallel bodies of a convex body. This does not limit the range of geometric inequalities that we are able to prove. It does provide the means to establish precisely when equality holds.

For the proof of (1), let

$$J(K) = \int_0^{2\pi} A(P(K,\theta))\,d\theta,$$

for any convex body K in E^2 with inradius r. By the previous arguments, $J(K_\lambda)$ is concave on the interval $0 \leqq \lambda \leqq r$. Define

$$J'(K) = 2A(P)L(K).$$

Suppose P is an m-sided polygon with unit outer normals $u_1, u_2, \cdots, u_m$. Let l_i, $1 \leqq i \leqq m$, be the length of the ith edge of P. Substituting $K_1 = K_2 = P$ in (12), gives

$$A(P) = A(P,P) = \frac{1}{2}\sum_{i=1}^{m} h(P,u_i)l_i = \frac{1}{2}\sum_{i=1}^{m} l_i.$$

The length of the ith edge of $P(B,\theta)$ is independent of θ and, therefore, is also l_i.

For all $\varepsilon > 0$

$$\begin{aligned} P(K+\varepsilon K^*,\theta) &= P(K,\theta) + \varepsilon P(K^*,\theta) \\ &\supset P(K,\theta) + \varepsilon P(B,\theta). \end{aligned}$$

It follows from this set inclusion and the theorem of Section 2 that, for $0 < \lambda < r$,

$$\begin{aligned} \frac{d}{d\lambda} J(K_\lambda) &= 2\int_0^{2\pi} A(P(K_\lambda,\theta), P(K_\lambda^*,\theta))\,d\theta \\ &\geqq 2\int_0^{2\pi} A(P(K_\lambda,\theta), P(B,\theta))\,d\theta. \end{aligned} \tag{15}$$

The rotation of a vector u through an angle θ in the clockwise direction shall be denoted $u(\theta)$. Using (12), we obtain

$$\begin{aligned} 2\int_0^{2\pi} A(P(K_\lambda,\theta),P(B,\theta))d\theta &= \int_0^{2\pi}\sum_{i=1}^m h(K_\lambda,u_i(\theta))l_i d\theta \\ &= \sum_{i=1}^m l_i \int_0^{2\pi} h(K_\lambda,u_i(\theta))d\theta \\ &= \sum_{i=1}^m l_i L(K_\lambda) \\ &= L(K_\lambda)\left(\sum_{i=1}^m l_i\right) \\ &= 2A(P)L(K_\lambda) = J'(K_\lambda). \end{aligned}$$

It remains to determine when equality holds in (15). Let $l_i(\lambda,\theta)$ be the length of the ith edge of $P(K_\lambda,\theta)$, $1 \leqq i \leqq m$. Using (12) once again, yields

$$\begin{aligned} A(P(K_\lambda,\theta),P(K_\lambda^*,\theta)) &= \frac{1}{2}\sum_{i=1}^m h(K_\lambda^*,u_i(\theta))l_i(\lambda,\theta) \\ &\geqq \frac{1}{2}\sum_{i=1}^m h(B,u_i(\theta))l_i(\lambda,\theta) \\ &= A(P(K_\lambda,\theta),P(B,\theta)). \end{aligned}$$

Equality holds in (15) if and only if equality holds above for almost all θ, $0 \leqq \theta < 2\pi$. Therefore, equality holds in (15) if and only if $K_\lambda^* = B$. Combining Corollary 2 and (14), we have

$$J(K) \geqq J(K_0) + 2A(P)A(K) \geqq 2A(P)A(K).$$

This completes the proof of (1). Equality holds if and only if $K_\lambda^* = B$, $0 < \lambda < r$, and the kernel is a point. From Corollary 1, $K_\lambda^* = B$ if and only if $K = K_0 + rB$, hence equality holds in (1) if and only if K is a ball in E^2.

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Department of Mathematics
Saint Mary's College
Moraga, CA 94575 USA